Received: 16 August 2024 | Revised: 1 July 2025 | Accepted: 23 January 2026

DOI: 10.1002/asjc.70094

WILEY

RESEARCH ARTICLE

Performance improvement of discrete-time linear-quadratic regulators applied to uncertain linear systems using the Tikhonov regularization method

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Funding information
Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro, Grant/Award Number: SEI 260003/011831/2023; Coordenação de Aperfeicoamento de Pessoal de Nível Superior—Brasil (CAPES) Finance Code 001, CNPq BPP/PQ-Sr, Grant/Award Number: 313335/2022-2

Abstract

The linear-quadratic regulator (LQR) problem of optimal control of an uncertain discrete-time linear system (DTLS) is revisited in this paper from the perspective of Tikhonov regularization. We show that an optimally chosen regularization parameter reduces, compared to the classical LQR, the values of a scalar error function, as well as the cost function. The scalar regularization parameter can be calculated using a standard parameter choice method. Simulations confirm performance improvement when this regularized control signal is applied to a DTLS subject to a varying sampling rate.

KEYWORDS

linear quadratic regulators, optimal control, robust feedback controllers, Tikhonov regularization

1 | INTRODUCTION

Optimal control problems seek to calculate a control signal that causes a dynamic system to have a desired behavior while minimizing some performance criterion [16], and can be formulated in terms of dynamic programming [4]. For linear time-invariant (LTI) systems with no uncertainties, in which the cost to be minimized is a quadratic function of the state and the control, the well-known *linear-quadratic regulator (LQR)* is a closed-loop state feedback controller.

Discrete-time linear systems (DTLSs) are usually subject to various types of uncertainties, such as unmodeled dynamics, uncertain parameters, noisy measurements, and variable sampling rates, among many others. In many practical applications, LQR control is robust to small differences between the model used in the design process and the actual dynamical system. However, if there are time variations or significant deviations from the dynamical model, these factors must be taken into account in the control design; otherwise, the performance of the closed-loop control system could be severely compromised.

Regularization methods were originally developed to improve the condition of ill-posed inverse problems [11]. In control theory, regularization has been used to filter out noisy data in adaptive control [2], system identification

[9] and data-driven control [10, 26]. LQRs can also be designed using data obtained from external measurements. In this approach, regularization plays a critical role in ensuring certainty equivalence and robust stability, especially when dealing with noisy data [26].

Regularization methods have been used to a limited extent in the design of controllers for uncertain systems [19]. In [21] the authors propose the use of the Tikhonov regularization method in a DTLS subject to varying sampling rates. Applying the regularized control signal to the uncertain plant minimizes a scalar error function, achieving a closer approximation to the target state.

The present article extends the approach presented in [21] to the regularization of LQRs for uncertain DTLSs, in order to improve its performance relative to the classical LQR approach. We show that there exists a theoretical optimum regularization parameter that minimizes a scalar error function and that the use of a standard, possibly suboptimal, parameter choice method also reduces this error. The proposed use of regularization is very simple to implement and computationally cheap. The application to uncertain discrete-time linear control systems outperforms the standard LQR controller.

The paper is organized as follows. In Section 2 we present some preliminaries on the use of regularization in control systems. In Section 3 we develop the proposed regularized LQR method. In Section 4, we illustrate the benefits of the regularized controller through some computational experiments. In Section 5, we discuss some aspects of the application of the Tikhonov method in the present work. We also present a regularized control system using a ℓ_1 norm as a penalty term to enhance sparsity of the optimal control signal. General conclusions are presented in Section 6.

2 | PRELIMINARIES AND RELATED WORKS

2.1 | Tikhonov regularization

Ill-posed problems are those where small changes in the observed data can lead to results that significantly deviate from the expected solution. Such deviations may arise from various sources, including inaccurate measurements, rounding errors, unexpected delays, or unmodeled dynamics. To address these issues, regularization techniques are essential. Regularization modifies the original ill-posed problem into a nearby, more stable problem that is less sensitive to disturbances. This adjustment helps to produce a solution that is closer to what would be obtained in a noise-free scenario, compared to the exact but potentially unreliable solution of the original ill-posed problem [11, 24].

Of the many regularization methods, one of the most widely used is the Tikhonov method [24], also referred to as *ridge regression* in the statistical literature. This method consists of the addition of a scalar penalty term to an objective function of an optimization problem in order to improve its condition. The penalty term is an ℓ_2 norm of an expression depending on the decision variables. For example, the simplest form of the Tikhonov method applied to a discrete optimization problem specified as a linear algebraic system of equations $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ is the noisy data and $\mathbf{x} \in \mathbb{R}^n$ is the vector of unknowns, is as follows:

$$\min \|A\mathbf{x} - \mathbf{b}\|_2^2 + \lambda\|\mathbf{x}\|_2^2 \tag{1}$$

where $\lambda \in \mathbb{R}_+$ is the regularization parameter. Good choices of this scalar parameter aim to strike a balance between the error resulting from regularization and the error introduced by noise in the right-hand side vector $\mathbf{b}$.

Techniques to choose the regularization parameter λ include the Morozov discrepancy principle [18], the generalized cross-validation (GCV) function [13], and the L curve [15] are among the most commonly used methods.

The discrepancy principle depends on accurately estimating the norm of the noise vector $\delta = \|A\mathbf{x}^* - \mathbf{b}\|$, where $\mathbf{x}^*$ is the exact solution of the noise-free problem. The regularization parameter is selected to ensure that the norm of the residual vector is approximately equal to δ, as described below:

$$\|A\mathbf{x}_{filt} - \mathbf{b}\| = \tau\delta \tag{2}$$

where $\tau \geq 1$ is a user-specified scalar independent of δ and $\mathbf{x}_{filt}$ is the regularized solution of (1).

The GCV method involves selecting the regularization parameter that minimizes the following scalar functional:

$$GCV := \frac{\|A\mathbf{x}_{filt} - \mathbf{b}\|^2}{(m - \sum_{i=1}^{n}\phi_i)^2} \tag{3}$$

where ϕ_i, $i \in \{1, \dots, n\}$ are filtering factors which depend on the regularization method chosen (see details in [13]).

The L curve is a logarithmic plot of the norm of the regularized solution $\|\mathbf{x}_{filt}\|$ versus the norm of the regularized residue $\|A\mathbf{x}_{filt} - \mathbf{b}\|$. Typically, this curve exhibits an approximate “L” shape. The method involves selecting the regularization parameter at the corner of the L-shaped curve.

In [20], the authors propose an adaptive method for determining the Tikhonov regularization parameter. This method involves minimizing a specially chosen Lyapunov function using a gradient descent algorithm. In [3], a survey of different methods to calculate this parameter is presented. Other parameter choice methods can be found in [12, 22] among many other references.

2.2 | Regularization of linear control systems

In applications of dynamical control systems affected by the presence of some kind of disturbances, it is usual to design robust feedback controllers capable of reducing such disturbances. The application of the Tikhonov regularization method to ill-posed linear systems was discussed in [19] and other works by the same authors. We give a brief outline of the approach used in these papers.

Let the dynamical continuous-time linear system (CTLS) be given by:

$$\dot{\mathbf{x}} = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$
$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{4}$$

where $t \in [t_0, t_1] \subset [0, \infty)$, A and B are operators belonging to a Hilbert space (to generalize to an infinite-dimensional setting), $\mathbf{x} \in \mathbb{R}^n$ is the state vector and $\mathbf{u} \in \mathbb{R}^m$ is the control signal.

The solution of (4) is given by

$$\mathbf{x}(t) = \Psi(t - t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Psi(t - s)B(s)\mathbf{u}(s)\partial s \tag{5}$$

where $\Psi(t)$ is a C_0 semigroup (the state transition matrix in the finite-dimensional case).

The control system (4) is said to be *exactly* controllable over $[t_0, t_1]$ if for every $\mathbf{x}_\tau$, $\tau \in (t_0, t_1)$, there exists $\mathbf{u}(t)$ such that (5) satisfies the condition $\mathbf{x}(\tau) = \mathbf{x}_\tau$.

The control system (4) is said to be *approximately* controllable over $[t_0, t_1]$ if for every $\epsilon > 0$, $\mathbf{x}_\tau$, $\tau \in (t_0, t_1)$, there exists $\mathbf{u}(t)$ such that the solution (5) satisfies the condition [19]

$$\|\mathbf{x}(\tau) - \mathbf{x}_\tau\| < \epsilon$$

For a given t_0, $\tau > t_0$, let us define a control linear operator C as

$$C\mathbf{u} := \int_{t_0}^{\tau} \Psi(\tau - s)B(s)\mathbf{u}(s)\partial s \tag{6}$$

The system (4) is also said to be controllable if and only if there exists a bounded $\mathbf{u}(t)$ such that

$$C\mathbf{u} = \mathbf{v} \tag{7}$$

where $\mathbf{v} := \mathbf{x}_\tau - \Psi(\tau - t_0)\mathbf{x}_0$ [7, s. 4].

As stated in [19], approximately controllable linear systems exhibit ill-posed characteristics in the sense of Tikhonov. Regularization of (7) using the standard form of the Tikhonov method can be written as

$$(C^*C + \lambda I)\mathbf{u} = C^*\mathbf{v} \tag{8}$$

where C^* is the *adjoint* operator of C (see definition in [7]).

In [19], the authors regularize an approximately controllable infinite-dimensional linear system given by a controlled heat equation. In [21], the authors present a methodology to calculate robust feedback controllers applied to a DTLS arising from the discretization of (4):

$$\mathbf{x}_{k+1} = A_k\mathbf{x}_k + B_k\mathbf{u}_k \tag{9}$$

It is assumed in [21] that uncertainty arises from varying sampling rates, and the method proposed to deal with this is based on the use of a control signal regularized by the Tikhonov method:

$$\mathbf{u}_{k,\lambda} = (B_k^\top B_k + \lambda I)^{-1}B_k^\top(\mathbf{x}_{k+1,t} - A_k\mathbf{x}_k) \tag{10}$$

where $\mathbf{x}_{k+1,t}$ is the target state or noise-free state at the next iteration (see details in [21]). The choice of the regularization parameter aims to minimize a convex scalar error function, thus reducing the effects produced by the varying sampling rates.

Next, we mention some works related to the regularization of optimal controllers. The papers [1, 8] propose the regularization of optimization problems, including the design of optimal controllers for dynamical systems, modeled by partial differential equations, with constraints on the state and on the control.

The papers [10, 26], on data-driven control of linear systems, rely on the minimization of a cost functional using linear matrix inequality (LMI) methods. Regularization of the minimization problem plays an important role in promoting robust stability when the data is corrupted by noise [26].

This paper continues the work presented by [21], proposing the use of the Tikhonov regularization method in LQR design for DTLSs subject to model uncertainties. In contrast to the papers cited above, we propose direct regularization of the classical LQR design of discrete controllers, without conversion to an LMI problem.

2.3 | Linear-quadratic regulators

Given the discrete-time linear system (9), when a sequence of control signals $\{\mathbf{u}_k\}$, $k = 0, \dots, N-1$ is applied, a sequence of state vectors $\{\mathbf{x}_k\}$, $k = 0, \dots, N$ is generated.

The design of an optimal control policy applied to (9) seeks to find a sequence of control signals that minimize a quadratic cost function defined as

$$J(\{\mathbf{x}_k\},\{\mathbf{u}_k\}) = \tfrac{1}{2}\mathbf{x}_N^\top H\mathbf{x}_N + \sum_{k=0}^{N-1}\tfrac{1}{2}\left[\mathbf{x}_k^\top Q\mathbf{x}_k + \mathbf{u}_k^\top R\mathbf{u}_k\right] \quad (11)$$

where $Q = Q^\top \geq 0$, $H = H^\top \geq 0$ and $R = R^\top > 0$ are weighting matrices chosen in the design process and N is the control horizon. The optimal control problem can be formulated as the following quadratic programming problem:

$$\begin{aligned} \min \quad & J(\{\mathbf{x}_k\},\{\mathbf{u}_k\}) \\ \text{s.t.} \quad & \mathbf{x}_{k+1} = A_k\mathbf{x}_k + B_k\mathbf{u}_k \quad k = 0,\ \dots\ ,N-1 \end{aligned} \quad (12)$$

if there are no constraints on the control signals or on the states. The time-invariant case, where A and B are constant matrices, will be considered in this section, but it can be easily extended to the time-varying case.

As $\mathbf{x}_k = A^k\mathbf{x}_0 + \sum_{j=0}^{k-1}A^{k-1-j}B\mathbf{u}_j$, $k = 1,\ \dots\ ,N$, the equality constraint in (12) can be eliminated to yield an unconstrained minimization problem. Defining $\mathbf{u} := [\mathbf{u}_0^\top\ \mathbf{u}_1^\top\ \dots\ \mathbf{u}_{N-1}^\top]^\top \in \mathbb{R}^{Nm}$ and $\mathbf{u}^*(\mathbf{x}_0) = \arg\min J(\mathbf{x}_0,\mathbf{u})$, the application of the control sequence $\mathbf{u}^*$ to the system (9) yields an open-loop control system.

In order to design a closed-loop or feedback optimal control $\mathbf{u}_k^* = \mathbf{f}(\mathbf{x}_k, k)$, Bellman's dynamic programming method can be applied [16]. The optimal feedback control is given by

$$\begin{aligned} \mathbf{u}_k^* &= -(R + B^\top P_{k+1}B)^{-1}B^\top P_{k+1}A\mathbf{x}_k, \\ k &= 0,\cdots,N-1 \end{aligned} \quad (13)$$

where the symmetric positive definite matrix P_k is given by

$$P_k = Q + A^\top\left[P_{k+1} - P_{k+1}B(R + B^\top P_{k+1}B)^{-1}B^\top P_{k+1}\right]A \quad (14)$$

calculated backward from $P_N = H$. The minimum value of the cost function (12) is given by

$$J^*(\mathbf{x}_0,\{\mathbf{u}_k^*\}) = \frac{1}{2}\mathbf{x}_0^\top P_0\mathbf{x}_0 \quad (15)$$

In the case of an infinite control horizon, P is a constant matrix which is the unique positive definite solution of the discrete algebraic Riccati equation (DARE)

$$P = Q + A^\top\left[P - PB(R + B^\top PB)^{-1}B^\top P\right]A \quad (16)$$

3 | OPTIMAL CONTROL REGULARIZED USING THE TIKHONOV METHOD

In this section, we consider the matrices A_k and B_k in (9) with entries subject to unknown nonconstant bounded uncertainties. However, their nominal values, denoted as $\bar{A}$ and $\bar{B}$ respectively, are considered to be known and constant.

For example, consider the uncertainties that arise from the discretization of a time-invariant CTLS with a varying sampling rate by a ZOH circuit (as in [21]). The sampling interval is denoted as $h_k = \bar{h} + \Delta h_k$, where $\bar{h}$ is the nominal value and Δh_k is a sampling jitter, assumed to be time-varying. In this case $A_k = e^{Ah_k} = e^{A\bar{h}}e^{A\Delta h_k} = \bar{A}e^{A\Delta h_k}$ and $B_k = \int_0^{h_k}e^{A\tau}\partial\tau B = \int_0^{\bar{h}}e^{A\tau}\partial\tau B + \int_{\bar{h}}^{h_k}e^{A\tau}\partial\tau B = \bar{B} + \int_0^{\Delta h_k}e^{A(\tau+\bar{h})}\partial\tau B$, where A and B are the constant matrices of the CTLS.

The LQR applied to the DTLS (9) is calculated using the nominal values of the matrices A_k and B_k, hence

$$\mathbf{u}_k^* = -(R + \bar{B}^\top P_{k+1}\bar{B})^{-1}\bar{B}^\top P_{k+1}\bar{A}\mathbf{x}_k \quad (17)$$

where P_{k+1} is calculated as in (14) in the finite horizon case, and $P_{k+1} = P$ is the constant solution of the DARE (16) in the infinite horizon case. When the control signal (17) is applied to the actual DTLS, it produces at the next iteration the state vector

$$\begin{aligned} \mathbf{x}_{k+1} &= A_k\mathbf{x}_k + B_k\mathbf{u}_k^* \\ &= [A_k - B_k(R + \bar{B}^\top P_{k+1}\bar{B})^{-1}\bar{B}P_{k+1}\bar{A}]\mathbf{x}_k \end{aligned} \quad (18)$$

The target state is defined as the one that would be obtained by application of (17) to the nominal system (i.e., without disturbances):

$$\mathbf{x}_{k+1,t} := [\bar{A} - \bar{B}(R + \bar{B}^\top P_k\bar{B})^{-1}\bar{B}^\top P_k\bar{A}]\mathbf{x}_k \quad (19)$$

The presence of disturbances in the matrices A and B produces a deviation from the target state. Thus, we propose the application of a regularized control signal to reduce this error. Tikhonov regularization can be implemented by adding an ℓ_2 norm penalty term to the cost function, that is:

$$J_\lambda = \frac{1}{2}\mathbf{x}_N^\top H\mathbf{x}_N + \sum_{k=0}^{N-1}\frac{1}{2}\left[\mathbf{x}_k^\top Q\mathbf{x}_k + \mathbf{u}_k^\top R\mathbf{u}_k + \lambda\|\mathbf{u}_k\|_2^2\right] \quad (20)$$

where $\lambda \in \mathbb{R}$ is the scalar regularization parameter (ridge parameter) to be chosen suitably in what follows.

Applying Bellman's dynamic programming principle, minimization of (20) yields

$$\mathbf{u}_{k,\lambda} = -(\lambda I + R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A}\mathbf{x}_k \tag{21}$$

The application of (21) to the perturbed system (9) produces the regularized state vector:

$$\mathbf{x}_{k+1,\lambda} = [A_k - B_k(\lambda I + R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A}]\mathbf{x}_k. \tag{22}$$

The goal is to make the regularized state (22) closer to the target state (19) than the unregularized state (18) at each iteration for a conveniently chosen regularization parameter. This can be formalized as choosing λ to meet the requirement:

$$\|\mathbf{x}_{k+1,t} - \mathbf{x}_{k+1,\lambda}\| < \|\mathbf{x}_{k+1,t} - \mathbf{x}_{k+1}\|$$

Defining the error function at the iteration k as

$$e_k(\lambda) := \|\mathbf{x}_{k+1,t} - \mathbf{x}_{k+1,\lambda}\| \tag{23}$$

the square of this error is calculated as:

$$\begin{aligned}
e_k(\lambda)^2 &= \|[\bar{A} - \overline{B}(R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A}]\mathbf{x}_k \\
&\quad - [A_k - B_k(\lambda I + R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A}]\mathbf{x}_k\|^2 \\
&= \|(\bar{A} - A_k - \overline{B}(R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A})\mathbf{x}_k \\
&\quad - B_k(\lambda I + R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A}\mathbf{x}_k\|^2 \\
&= \|(\bar{A} - A_k - \overline{B}(R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A})\mathbf{x}_k\|^2 \\
&\quad + 2\mathbf{x}_k^\top(\bar{A} - A_k - \overline{B}(R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A})^\top \\
&\quad \times B_k(\lambda I + R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A}\mathbf{x}_k \\
&\quad + \|B_k(\lambda I + R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A}\mathbf{x}_k\|^2
\end{aligned}$$

This is a quadratic convex function with a minimum given by the value of λ, which zeroes the derivative:

$$\begin{aligned}
\frac{\partial e_k^2}{\partial \lambda} &= -2\mathbf{x}_k^\top \bar{A}^\top P_{k+1}\overline{B}(\lambda I + R + \overline{B}^\top P_{k+1}\overline{B})^{-2} \\
&\quad \times \Big[B_k^\top(\bar{A} - A_k - \overline{B}(R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A}) \\
&\quad + B_k^\top B_k(\lambda I + R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A}\Big]\mathbf{x}_k
\end{aligned}$$

The optimal value λ_k^*, which minimizes (23), is a nonlinear regression problem that can be solved using a gradient descent algorithm.

If $\lambda = 0$, then $e_k = \|\mathbf{x}_{k+1,t} - \mathbf{x}_{k+1}\|$, which corresponds to the error introduced by the disturbances in the entries of the matrices. Note that if there is no disturbance, that is $A_k = \bar{A}$ and $B_k = \overline{B}$, the deviation from the target state is due to regularization. Then

$$e_k = \|B\big[(R + B^\top P_{k+1}B)^{-1} - (\lambda I + R + B^\top P_{k+1}B)^{-1}\big]B^\top P_{k+1}A\mathbf{x}_k\|,$$

which, in turn, is obviously zero when $\lambda = 0$. Note also that in this case $\lambda_k^* = 0$ is the optimal value of λ which zeros the derivative of the error. The optimal value of λ is a tradeoff between the error produced by noise and the deviation from the target state produced by the addition of the penalty term, thus minimizing (23).

Of course, since A_k and B_k are not previously known, it is not possible to calculate this optimal value in practice. Other methods of choosing the regularization parameter, as those mentioned in Section 2.1, should be used.

The application of the optimal control signal (17) (calculated for the nominal system) to the real system is, of course, suboptimal and will result in increased cost. If the regularized control signal (21) is applied instead, the cost function can be reduced by an adequate choice of the regularization parameter.

According to Bellman's dynamic programming principle, the optimal cost function to go from the iteration k, $k \in \{0, \dots, N-1\}$, to the final iteration N can be expressed as

$$\begin{aligned}
J_k^*\left(\mathbf{x}_k, \{\mathbf{u}\}_k^{N-1}\right) &= \frac{1}{2}\mathbf{x}_{k+1}^\top P_{k+1}\mathbf{x}_{k+1} \\
&\quad + \frac{1}{2}\mathbf{x}_k^\top Q\mathbf{x}_k + \frac{1}{2}\mathbf{u}_k^\top R\mathbf{u}_k
\end{aligned} \tag{24}$$

The application of the regularized control (21) to the actual system produces a *cost-to-go* function

$$\begin{aligned}
J_{k,\lambda} &= \frac{1}{2}(A_k\mathbf{x}_k + B_k\mathbf{u}_{k,\lambda})^\top P_{k+1}(A_k\mathbf{x}_k + B_k\mathbf{u}_{k,\lambda}) \\
&\quad + \frac{1}{2}\mathbf{x}_k^\top Q\mathbf{x}_k + \frac{1}{2}\mathbf{u}_{k,\lambda}^\top R\mathbf{u}_{k,\lambda} \\
&= \frac{1}{2}\mathbf{x}_k^\top\Big[A_k^\top P_{k+1}A_k + Q \\
&\quad - 2A_k^\top P_{k+1}B_k(\lambda I + R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A} \\
&\quad + \bar{A}^\top P_{k+1}\overline{B}(\lambda I + R + \overline{B}^\top P_{k+1}\overline{B})^{-1}(R + B_k^\top P_{k+1}B_k) \\
&\quad \times (\lambda I + R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A}\Big]\mathbf{x}_k
\end{aligned} \tag{25}$$

This is a quadratic convex function with a minimum given by the value of λ, which zeroes its derivative, that is,

$$\begin{aligned}
\frac{\partial J_{k,\lambda}}{\partial \lambda} &= \mathbf{x}_k^\top \bar{A}^\top P_{k+1}\overline{B}(\lambda I + R + \overline{B}^\top P_{k+1}\overline{B})^{-2} \\
&\quad \times \Big[B_k^\top P_{k+1}A_k - (R + B_k^\top P_{k+1}B_k) \\
&\quad \times(\lambda I + R + \overline{B}^\top P_{k+1}\overline{B})^{-1}\overline{B}^\top P_{k+1}\bar{A}\Big]\mathbf{x}_k
\end{aligned}$$

The optimal value $\lambda_k^* = \arg\min J_{k,\lambda}$, which minimizes (25), can be calculated via nonlinear regression using a gradient descent algorithm. Note that the value $\lambda = 0$ yields the *cost-to-go* function which would be obtained if the unregularized control signal (17) were applied to the actual system. Note also that if there is no disturbance, that is, $A_k = \bar{A}$ and $B_k = \overline{B}$ for all k, increase of the cost function, with respect to the nominal case, is due to the regularization term in the objective function. In this case, the optimal value of $\lambda_k^* = 0$ because it minimizes (25), zeroing its derivative.

The results above are summarized in the following lemma.

Lemma 3.1. *Consider a discrete-time linear system (9) subject to uncertainties in the matrices A_k and B_k. If at an iteration k an optimal control signal calculated from the minimization of a scalar cost function is applied, at the next iteration the system produces a state vector different from the desired target state (19), modifying also the optimal cost-to-go function (24). The error is quantified as $e_k(0)$ in (23). If a control signal $\mathbf{u}_{k,\lambda}$ calculated as in (21) is applied at each iteration, with a value of the regularization parameter λ between 0 and λ_k^* (the minimizer of $e_k(\lambda)$ or the minimizer of $J_{k,\lambda}$, the scalar error and the cost-to-go function are reduced, reaching their minimum at their respective values λ_k^*.* □

3.1 | Illustrative example of convexity of the error and cost-to-go functions

Figure 1 shows the error function $e_k(\lambda)$ vs. λ with the system that will be exemplified in the following section. This error is produced at the first iteration by an inaccurate sampling interval $h_0 = \overline{h} + \Delta h_0 = 1 + 0.47\,s$. The optimal value of the regularization parameter is $\lambda_k^* = 0.2126$, and the error of the unregularized state vector is $e_k(0) = 0.49$.

Figure 2 shows the *cost to-go* function with the system that will be used in the following section at the first iteration, where the disturbance is produced by an uncertainty in the sampling rate equal to $\Delta h_k = -0.2\ s$.

4 | COMPUTATIONAL EXPERIMENTS

In this section, we present the results of the application of the control signal (21) to a DTLS.

We exemplify our theoretical results via a simulation case study. We consider the system used in [21]. This is a dynamic model of a paper machine head box. Let the dynamical CTLS be given by:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t)\\ &= \begin{bmatrix} -0.2 & 0.1 & 1 \\ -0.05 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0.7 \\ 1 & 0 \end{bmatrix}\mathbf{u}(t)\end{aligned} \tag{26}$$

The system described in (26) exhibits stability, with eigenvalues $\sigma \in \{-0.1707, -0.0293, -1\}$. This system is sampled by a ZOH circuit at a varying rate with a Gaussian distribution, characterized by a mean of $\overline{h} = 1$ second and a standard deviation of 0.4 seconds. The resulting DTLS, sampled at the nominal rate $\overline{h}$, remains stable, with the matrix $\bar{A}$ possessing eigenvalues $\sigma \in \{0.8431, 0.9711, 0.3679\}$. The nominal DTLS is

$$\begin{aligned}\mathbf{x}_{k+1} &= \bar{A}\mathbf{x}_k + \overline{B}\mathbf{u}_k\\ &= \begin{bmatrix} 0.8165 & 0.0906 & 0.5630 \\ -0.0453 & 0.9977 & -0.0171 \\ 0 & 0 & 0.3679 \end{bmatrix}\mathbf{x}_k\\ &+ \begin{bmatrix} 0.3426 & 0.9384 \\ -0.0063 & 0.6760 \\ 0.6321 & 0 \end{bmatrix}\mathbf{u}_k\end{aligned} \tag{27}$$

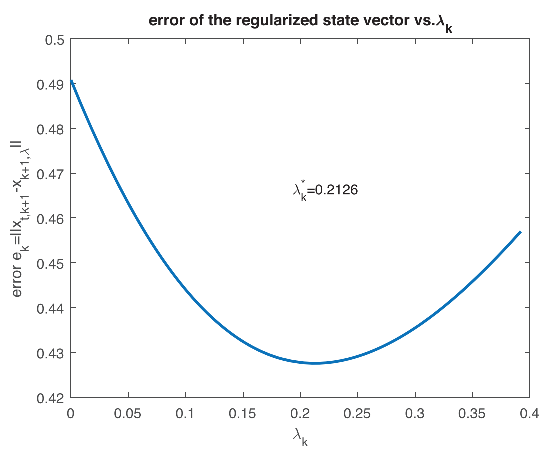

FIGURE 1 Error function vs. regularization parameter.

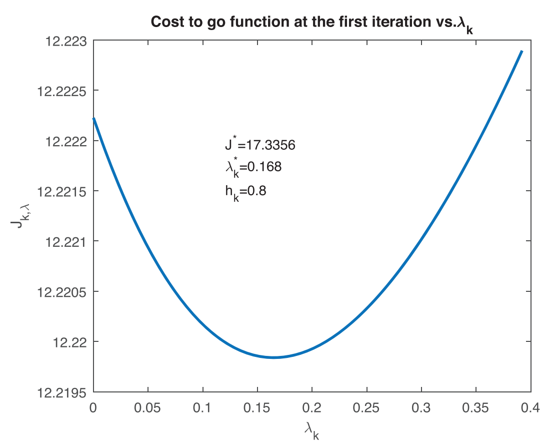

FIGURE 2 Cost-to-go function vs. regularization parameter.

Simulations were carried out over a duration of 10 seconds, so the number of iterations $N = 10$. The sampling intervals are: $h_k \in \{1.4696s, 1.0508s, 0.7373s, 0.4074s, 1.0622s, 1.3274s, 0.8830s, 0.7837s, 0.8765s, 0.5614s, \}$.

The initial state was chosen as $\mathbf{x}_0 = [2\ -3\ -1]^{\top}$.

We consider the cost function (11) with matrices $H = Q = I$ and $R = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.5 \end{bmatrix}$. With these values, the optimal cost function $J^* = 17.3356$.

The matrices $P_k, k \in \{0, \dots, 10\}$, were calculated as in (14) from $P_{10} = H$. The regularization parameter was chosen as a constant $\lambda = 0.034$. The regularized control signals were calculated as in (21), whereas the unregularized control signals were calculated as in (17). Both of them use the nominal matrices shown in (27). The matrix inversion in (17) and (21) are calculated using QR factorization.

Figure 3 shows the components of the regularized control signal vs. time, where the different sampling intervals at which the control signal is applied at each iteration can be seen.

Figure 4 shows the components of the target state vector, the components of the state vector resulting from the application of the unregularized control signal (17) to the CTLS, and the state vector produced by the application of the regularized control signal (21) to the CTLS vs. time.

We define the average error accumulated over all iterations as:

$$\overline{\epsilon}(\lambda) := \frac{\sum_{k=0}^{N-1} \epsilon_k(\lambda)}{N} = \frac{\sum_{k=0}^{N-1} \|\mathbf{x}_{k+1,\lambda} - \mathbf{x}_{k+1,t}\|}{N} \tag{28}$$

Figure 5 shows the relationship between the average error (28) and the regularization parameter λ (assuming the use of a constant λ for all iterations). The curve reaches its minimum value at $\lambda^* = 0.034$. As discussed in Section 3,

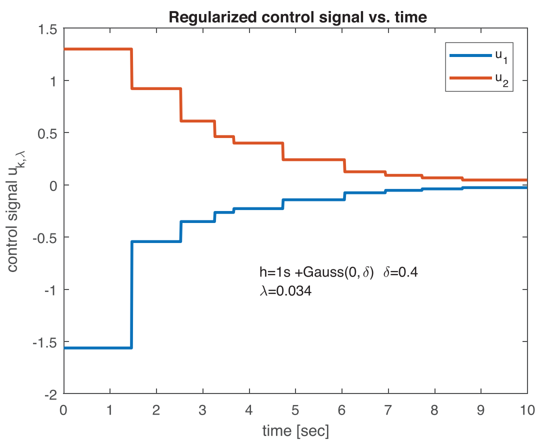

FIGURE 3 Regularized control signal vs. time.

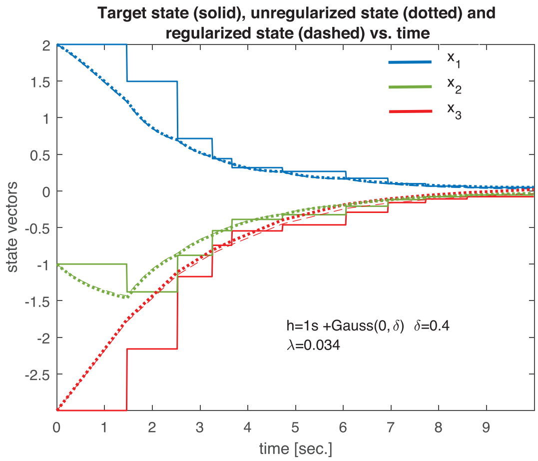

FIGURE 4 Time plot of state vector components. The solid lines are the components of the discrete target state. The dotted lines represent the components of the continuous-time plant when the unregularized control signal is applied. The dashed lines represent the state components of the continuous-time plant when the regularized control signal is applied.

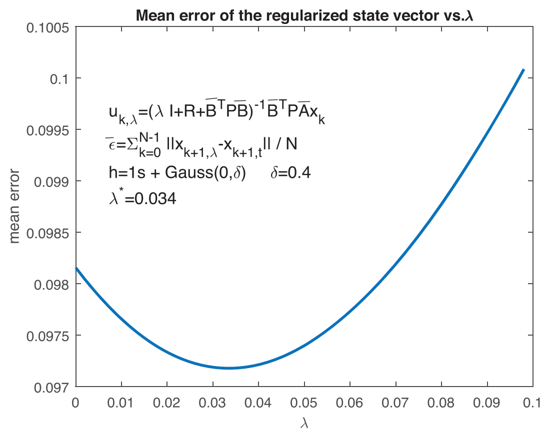

FIGURE 5 Mean error given by Equation (28) vs. the regularization parameter λ.

when $\lambda = 0$, the control input $\mathbf{u}_{k,\lambda}$ equals $\mathbf{u}_k$, and the state vector $\mathbf{x}_{k+1,\lambda}$ equals $\mathbf{x}_{k+1}$. Consequently, the error $\epsilon_k(0)$ calculated as in (23), as well as the mean error $\overline{\epsilon}(0)$, match those of the unregularized state vector. Indeed, the regularized cost-to-go function $J_{k,0}$ calculated as in (25) also equals the value of the unregularized cost-to-go function (24) at each iteration.

4.1 | Choice of the regularization parameter

This section discusses the important question of the choice of the regularization parameter. As discussed in Section 3, utilizing the value of λ_k which minimizes (23) or (25) at each iteration is not feasible in practical scenarios due to the unknown values of A_k and B_k. Various methods mentioned in Section 2.1 were explored (detailed descriptions of these methods can be found in [13, 15, 18], respectively). Specifically, the minimum of the GCV function was determined using the gcv function, the corner of the L curve was identified using the l_curve function, and the value of λ satisfying the discrepancy principle was found using the discrep function. These functions are all part of Hansen's regularization toolbox for MATLAB [14]. Since the right-hand side of the equation $(\lambda I + R + \overline{B}^{\top} P_{k+1}\overline{B})\mathbf{u}_{k,\lambda} = -\overline{B}^{\top} P_{k+1}\bar{A}\mathbf{x}_k$ varies at each iteration, these functions calculate a different value of λ at each iteration, as opposed to a constant value as that used in the test shown in Figure 5.

This leads to the development of an adaptive gain controller that outperforms a static state feedback controller (see details in [21]). Table 1 presents the mean error and the cost function for different constant values of λ, as well as the mean error when this parameter is determined by the three aforementioned functions. The theoretically calculated parameters $\lambda^*_{k_1}$ computed as the minimizer of (23) and $\lambda^*_{k_2}$ computed as the minimizer of (25) at each iteration were also tested. These optimal values were found at each iteration using the function fminsearch of MATLAB. It is worth noting that, while the unregularized state vector remains independent of λ, the target state vector depends on it. This dependency arises because the target state vector is computed using the regularized state from the previous iteration, resulting in the mean error of the unregularized system also being influenced by λ.

In this experiment, the functions l_curve and gcv do not find the corner of the L curve and the minimum of the cross-validation function, respectively. The function discrep found a value of λ that satisfactorily regularized the control signal, producing an error value not much greater than its minimum value (achieved with the theoretical optimal regularization parameter) and a cost function slightly greater than its minimum value. The discrep function requires a reliable estimate of the magnitude of the error [14, 18]. Note also that the minimum regularized mean error is produced when a value $\lambda^*_{k_1}$ is used at each iteration, as well as the minimum regularized cost function is produced when $\lambda^*_{k_2}$ is used, as expected.

We repeated the experiment using a constant matrix P solution of the discrete algebraic Riccati Equation (16), but the results are omitted here because they were not significantly different from those reported in the first experiment.

5 | DISCUSSION AND FUTURE WORKS

In inverse problems, it is well known that the minimization of a ℓ_2 norm of an error function leads to a smooth dense least squares solution very sensitive to outliers [5, c. 3]. In underdetermined discrete constrained optimization problems of the form

$$\min \|\mathbf{x}\|_p, \quad \text{s.t.} \|A\mathbf{x} - \mathbf{b}\|_2 = 0$$

the choice of $p = 1$, which is a nondifferentiable ℓ_1 convex minimization problem, leads to a sparse solution potentially more robust to outliers and corrupt data. A related convex optimization problem is given by:

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2 + \lambda\|\mathbf{x}\|_1 \tag{29}$$

where $\lambda \in \mathbb{R}_+$ is the regularization parameter that weights the importance of sparsity.

The problem (29) is known as the *least absolute shrinkage and selection operator* (LASSO), first introduced in [23]. This is a ℓ_1 penalized regression technique that balances model complexity with descriptive capability and performs well even for overdetermined problems. LASSO may be thought of as a sparsity-promoting regression that benefits from the stability of the ℓ_2 regularized ridge regression (see [5, c. 3–4] and references therein).

In optimal control problems, regularization of the standard quadratic performance index with an ℓ_1 penalty term promotes sparsity on the solution. In problems of medium

TABLE 1 Mean error of the regularized state vector $\bar{e}_r = \sum_{k=0}^{N-1} \|\mathbf{x}_{k+1,\lambda} - \mathbf{x}_{k+1,t}\|/N$, mean error of the unregularized state vector $\bar{e}_u = \sum_{k=0}^{N-1} \|\mathbf{x}_{k+1} - \mathbf{x}_{k+1,t}\|/N$, regularized cost function J_r and unregularized cost function J_u for different values of the parameter λ using a matrix P_k calculated as in (14). $\lambda^*_{k_1} = \arg\min e_k(\lambda)$, $\lambda^*_{k_2} = \arg\min J_{k,\lambda}$.

λ	0	0.01	0.034	0.06	0.1	$\lambda^*_{k_1}$	$\lambda^*_{k_2}$	l_curve	gcv	discrep
$\bar{e}_r$	0.0977	0.0972	0.0967	0.0972	0.0996	0.0938	0.1084	0.4992	0.9302	0.1191
$\bar{e}_u$	0.0977	0.0979	0.0985	0.0992	0.1002	0.1011	0.0968	0.1636	0.2376	0.1218
J_r	14.8908	14.9003	14.9253	14.9550	15.0062	15.1293	14.8159	19.7463	27.3600	15.5611
J_u	14.8908	14.9310	15.0276	15.1320	15.2923	15.4875	14.9059	21.7743	28.3773	16.3964

to large dimensions, it may be interesting to obtain sparse control signals, when some of their components may not be applied in all iterations. In [17], a sparse feedback gain of an optimal controller for CTLS is designed.

With a quadratic cost functional, the implementation of the LASSO method to promote sparsity of the solution can be expressed as

$$\begin{aligned} &\min\ J(\{\mathbf{x}_k\},\{\mathbf{u}_k\}) \\ &\quad = \tfrac{1}{2}\mathbf{x}_N^\top H\mathbf{x}_N + \sum_{k=0}^{N-1}\tfrac{1}{2}\left[\mathbf{x}_k^\top Q\mathbf{x}_k + \mathbf{u}_k^\top R\mathbf{u}_k\right] + \lambda\|\mathbf{u}_k\|_1 \\ &\text{s.t}\quad \mathbf{x}_{k+1} = A_k\mathbf{x}_k + B_k\mathbf{u}_k, \quad k = 0, \dots, N-1 \end{aligned} \tag{30}$$

The minimization of the cost functional (30) has been implemented with the dynamical system described in Section 4. The matrices A_k and B_k, $k = 0, \dots, N-1$ were the same as those used in the former experiment, as well as the weighting matrices used in the cost function, the control horizon, and the initial state. Figure 6 shows the mean error over 10 iterations vs. the regularization parameter λ.

Figure 6 shows a convex function, where an optimal regularization parameter minimizes the mean error over all the iterations. In this case, this optimal regularization parameter was graphically determined as $\lambda^* = 0.2843$. Of course, this result is also theoretical because the noisy matrices are not previously known.

Figure 7 shows the cost function (11) vs. the regularization parameter. The control signal applied to the actual system is the one that minimizes (30). The optimal parameter was also graphically determined as $\lambda^* = 0.3265$.

The resulting control vector $\mathbf{u}^*$, minimum argument of (30), exhibits certain sparsity whatever the value of λ used.

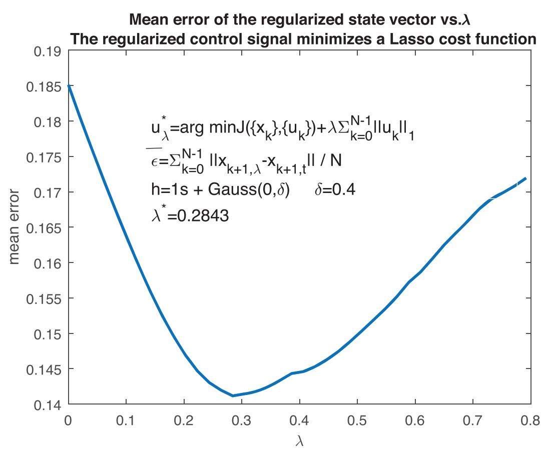

FIGURE 6 Mean error given by Equation (28) vs. the regularization parameter λ where the regularized control signal is the minimizer of (30).

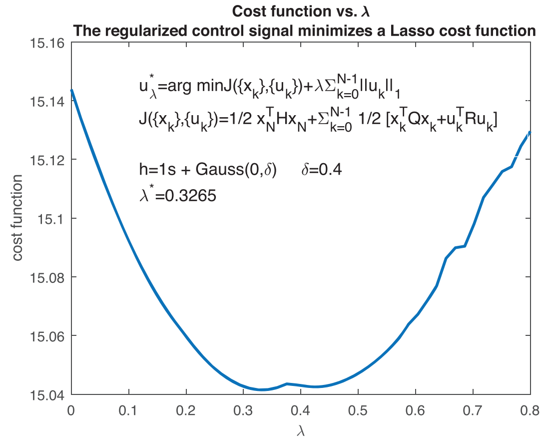

FIGURE 7 Cost function given by Equation (11) vs. the regularization parameter λ where the regularized control signal is the minimizer of (30).

The application of this control signal to the DTLS (9) yields an open-loop control system.

To design a closed-loop feedback controller, a recursive form of the LASSO method should be used, in such a way that at each iteration, the optimum control signal $\mathbf{u}_k^*$ that minimizes a cost-to-go function would be calculated. A recursive LASSO algorithm is proposed in [6]. In [25], a continuous-time tracking control regularized in order to promote sparsity of the solution is proposed.

In future work, we propose to research the possibility of implementing a feedback controller that produces a sparse control signal calculated at each iteration.

6 | CONCLUSIONS

The regularized optimal feedback control (21) has shown the ability to reduce a scalar error function as well as the cost function when applied to a DTLS subject to model uncertainties along N iterations. Table 1 clearly shows the reduction of the mean error and of the cost function when the regularized control signal is applied with respect to the application of the classical LQR.

The reduction of the error strongly depends on the choice of the scalar regularization parameter. Among the standard parameter choice methods, the search for the corner of the L curve and the minimization of the GCV function did not present good results in the experiment reported here. The application of Morozov's discrepancy principle seems to be the best option for choosing the regularization parameter. Using the discrepancy principle requires that a reliable bound of the error norm be available. If such a bound is not available, then the choice of a good enough regularization parameter is still an open issue, and more research should be devoted to this end.

AUTHOR CONTRIBUTIONS

Fernando Pazos: Conceptualization; data curation; formal analysis; funding acquisition; investigation; methodology; software; validation; visualization; writing – original draft; writing – review and editing. **Amit Bhaya:** Conceptualization; data curation; formal analysis; funding acquisition; investigation; methodology; software; validation; visualization; writing – original draft; writing – review and editing.

FUNDING INFORMATION

Fernando Pazos' work was supported by FAPERJ—Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro, Processo SEI 260003/011831/2023 as a post-doctoral researcher in the Dept of Electrical Engineering, COPPE/UFRJ.

Amit Bhaya's work was supported by Coordenação de Aperfeicoamento de Pessoal de Nível Superior—Brasil (CAPES) Finance Code 001, CNPq BPP/PQ-Sr grant no. 313335/2022-2.

CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

DATA AVAILABILITY STATEMENT

No new data generated.

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How to cite this article: F. Pazos and A. Bhaya, *Performance improvement of discrete-time linear-quadratic regulators applied to uncertain linear systems using the Tikhonov regularization method*, Asian J. Control. (2026), 1–11. https://doi.org/10.1002/asjc.70094